A Comprehensive Review of the ac-NEGF and Floquet-NEGF Techniques

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The non-equilibrium Green's function (NEGF) formalism has emerged as a powerful tool for describing quantum transport phenomena in novel nanomaterials and devices. The ac-NEGF and Floquet-NEGF methods, in particular, have been instrumental in understanding and predicting the behaviour of systems driven out of equilibrium by time-dependent fields. In this article, I have presented a brief overview of the aforementioned methods, primarily focusing on their mathematical formulation and concluding with a note on the fundamental differences that set these two methods apart.

I. INTRODUCTION

In classical physics, Green functions are used as a powerful method for solving inhomogeneous differential equations. Many-body physics Green functions turn out to be a very powerful technique for evaluating the properties of such systems in both thermal equilibrium and nonequilibrium situations.

The formulations for the retarded, advanced, and the "lesser" and "greater" Green functions used henceforth are given by

$$\begin{aligned}
G^r(r,t;r',t') &= -i\theta(t-t')\langle\{\psi(r,t),\psi^\dagger(r',t')\}\rangle,\\
G^a(r,t;r',t') &= i\theta(t'-t)\langle\{\psi(r,t),\psi^\dagger(r',t')\}\rangle,\\
G^<(r,t;r',t') &= i\langle\psi^\dagger(r',t')\ \psi(r,t)\rangle,\\
G^>(r,t;r',t') &= i\langle\psi(r,t)\ \psi^\dagger(r',t')\rangle
\end{aligned}$$

where $\theta(t)$ represents the Heaviside step function. These various functions are not independent; they obey

$$G^r - G^a = G^> - G^<$$

The time-ordered, the retarded, and the advanced Green functions can be expressed in terms of $G^>$ and $G^<$:

$$\begin{aligned}
G(r,t;r',t') &= \theta(t-t')G^> + \theta(t'-t)G^<,\\
G^{r,a}(r,t;r't') &= \pm\theta(\pm t \mp t')[G^> - G^<]
\end{aligned}$$

Each formulation has its own advantages. For example, $G^{<,>}(r,t;r',t')$ are directly linked to observables, while $G^{r,a}(r,t;r',t')$ are well suited for determining the physical response of a system.

II. AC-NEGF FORMALISM

As described in Ref.[1], the authors have developed a linear response theory for self-consistent ac quantum transport employing nonequilibrium Green functions solved self-consistently with Poisson's equation in the presence of a time-dependent potential at a non-transport terminal. Unlike source-drain contacts, which are involved in charge transport, nontransport terminals are coupled to the device channel through the dynamic potential. Further, this approach was applied to a carbon nanotube field-effect transistor (NTFET) in order to determine its high-frequency response.

A. Model Hamiltonian

Let us begin by specifying the Hamiltonian operator of the system. The total system can be divided into three isolated regions, and the Hamiltonian of the entire infinite system is written as

$$H = H_d + H_c + H_t \tag{1}$$

where H_d is the device Hamiltonian, H_c refers to the two semi-infinite leads, and H_t couples the device region to the leads. The device Hamiltonian can be further simplified as

$$H_d = H_d^0 + H_d^{DC} + H_d^{AC} \tag{2}$$

where

$$H_d^0 = \sum_n \epsilon_n^0 \hat{c}_n^\dagger \hat{c}_n + \sum_{n,m} t_{n,m}\hat{c}_n^\dagger \hat{c}_m + \text{h.c.}, \tag{3}$$

and

$$H_d^{\text{DC}} = \sum_n U_n^{\text{DC}} \hat{c}_n^\dagger \hat{c}_n, \quad H_d^{\text{AC}} = \sum_n U_n(t)\hat{c}_n^\dagger \hat{c}_n, \tag{4}$$

where $\hat{c}_n^\dagger$ and $\hat{c}_n$ refer to fermionic creation and annihilation operators at site n, respectively. The term U_n^{DC} represents a spatially-varying but time-independent electrostatic potential, which can be incorporated into the on-site energies ϵ_n^0. The problem under consideration originates from the presence of an *a priori unknown* time- and space-dependent potential $U_n(t)$ induced by externally applied time-dependent fields. Both U_n^{DC} and $U_n(t)$ must be determined separately by solving Poisson's equation in a self-consistent manner.

The Hamiltonian for the two contacts to the left and right ($\alpha = s, d$) of the device reads

$$H_c = \sum_{k,\alpha} \epsilon_{k\alpha}^0 \hat{c}_{k\alpha}^\dagger \hat{c}_{k\alpha} \tag{5}$$

where $\hat{c}^{\dagger}_{k\alpha}$ and $\hat{c}_{k\alpha}$ are fermionic creation and annihilation operators for a particle in terminal α in state k.

The Hamiltonian H_t couples the device subspace with the semi-infinite source and drain reservoirs.

$$H_t = \sum_{k\alpha,n} T_{n,k\alpha}\hat{c}^{\dagger}_{n}\hat{c}_{k\alpha} + T^{*}_{n,k\alpha}\hat{c}^{\dagger}_{k\alpha}\hat{c}_{n} \tag{6}$$

Note that Eq. (6) describes only the coupling between the device and *transport* terminals, but not to *non-transport* terminals.

B. Quantum dynamics and nonequilibrium statistics

The next step is to describe the carrier dynamics within the device scattering region using Green functions. We adopt a shorthand notation $G(t,t^{'})$ for Green functions, which are, in general, functions of both space and time, $G(\mathbf{r}t;\mathbf{r}^{'}t^{'})$. The time-evolution of the Green functions is governed by the Dyson equation

$$G^{\gamma}(t,t^{'}) = g^{\gamma}_{0}(t,t^{'}) + \int dt_1 dt_2\; g^{\gamma}_{0}(t,t_1)\Sigma^{\gamma}(t_1,t_2)G^{\gamma}(t_2,t^{'}), \tag{7}$$

where $g^{\gamma}_{0}(t,t^{'}) = g^{\gamma}_{0}(t-t^{'})$ refers to the retarded/advanced ($\gamma = \mathbf{r},\mathbf{a}$) Green function of the *isolated* system. The self-energy $\Sigma^{\gamma}(t,t^{'})$ accounts for all interactions of the isolated system with its environment. In this particular case under consideration, the self-energy $\Sigma^{\gamma}(t,t^{'})$ can be divided into three contributions

$$\Sigma^{\gamma}(t,t^{'}) = \sum_{\alpha=s,d}\Sigma^{\gamma}_{\alpha}(t-t^{'}) + U^{\mathrm{DC}}\delta(t-t^{'}) + U(t)\delta(t-t^{'}) \tag{8}$$

The first term $\Sigma_c = \sum_{\alpha=s,d}\Sigma_\alpha$ is the contact self-energy and connects the device region with the semi-infinite source and drain contacts. The second term is a scalar potential corresponding to the response of the device to externally applied time-independent fields. The third term is of particular interest to us since it describes the *dynamic* response of the device due to external time-dependent fields. Since the time-dependent signal is being applied at the gate (*nontransport*) terminal of the NTFET, the induced potential $U(t)$ distorts only the device scattering region while the contacts remain in a steady state.

In the energy domain, the self-energy, cf. Eq. (8) is given by

$$\Sigma^{\gamma}(E,E^{'}) = 2\pi\delta(E-E^{'})[\Sigma^{\gamma}_{c}(E)+U^{\mathrm{DC}}]+U(E-E^{'}) \tag{9}$$

This can be achieved by switching from the time domain into energy-domain representation via a double-time Fourier transform, more details about which have been presented in Appendix A.

Fourier transforming Eq. (7) and using Eq. (9), the effective Dyson equation for the device is given by

$$G^{\gamma}(E-E^{'}) = 2\pi\delta(E-E^{'})G^{\gamma}_{0}(E) + \int\frac{d\overline{E}}{2\pi}G^{\gamma}_{0}(E)U(E-\overline{E})G^{\gamma}(\overline{E},E^{'}), \tag{10}$$

where

$$G^{\gamma}_{0}(E) = [g^{\gamma}_{0}(E)^{-1} - U^{\mathrm{DC}} - \Sigma^{\gamma}_{c}(E)^{-1}] \tag{11}$$

and

$$g^{\gamma}_{0}(E) = [(E\pm i\eta)I - H^{0}_{d}]^{-1}, \tag{12}$$

with an infinitesimal $\eta > 0$. In the reformulated Dyson equation in Eq. (10), we have two distinct terms describing the dynamic response of the system. The first term describes the system's response in contact with the leads and subject to a dc electrostatic potential. The ac component, i.e., the second term, contains this term as well (G^{γ}_{0}) and determines the *distortion* of the system away from the ideal operation point and is driven by the time-dependent potential $U(t)$.

The deviation of total nonequilibrium particle distribution $G^{<}$ from its reference distribution at dc in the presence of the ac potential U is determined by mapping Dyson's equation for $G^{<}$, which gives: $G^{<} = G^{<}_{0} + G^{<}_{0}UG^{a} + G^{r}_{0}UG^{<}$. After the Fourier transform, the particle distribution is given by

$$\begin{aligned} G^{<}(E,E^{'}) &= 2\pi G^{<}_{0}(E)\delta(E-E^{'}) \\ &\quad + \int\frac{d\overline{E}}{2\pi}[G^{<}_{0}(E)U(E-\overline{E})G^{a}(\overline{E},E^{'}) \\ &\quad + G^{r}(E,\overline{E})U(\overline{E}-E^{'})G^{<}_{0}(E^{'})] \\ &\quad + \int\frac{dE_1}{2\pi}\frac{dE_2}{2\pi}\frac{dE_3}{2\pi}G^{r}(E,E_1)U(E_1-E_2)G^{<}_{0}(E_2) \\ &\quad \times U(E_2-E_3)G^{a}(E_3,E^{'}), \end{aligned} \tag{13}$$

where $G^{<}_{0}(E) = G^{r}_{0}(E)\Sigma^{<}_{c}(E)G^{a}_{0}(E)$ corresponds to the non-equilibrium spectral density at dc. The function $\Sigma^{<}_{c}(E) = \Sigma_\alpha i f_\alpha(E)\Gamma_\alpha(E)$ where $\Gamma_\alpha(E) = i(\Sigma^{r}_{\alpha} - \Sigma^{a}_{\alpha})$ is the broadening function, and $f_\alpha(E) = 1/[1 + e^{(E-\mu_\alpha)/k_BT}]$ is the Fermi function at temperature T with μ_α being the chemical potential of terminal α.

While the equations mentioned so far describe the quantum transport and the nonequilibrium statistics, we do not yet have a solution for the dynamic potential U. This must be obtained by solving Poisson's equation

$$\nabla[\epsilon(\mathbf{r})\nabla U(\mathbf{r},E-E^{'})] = -\rho(\mathbf{r},E-E^{'}), \tag{14}$$

with the frequency-dependent charge density

$$\rho(\omega) = ie\int\frac{dE}{2\pi}G^{<}(E^{+},E) \tag{15}$$

The calculation of the ac charge density requires $G^{<}$ to be evaluated at two energies $(E^{+},E) = (E+\hbar\omega,E)$, in contrast to the dc case where only one energy is needed. Furthermore, equations (14) and (15) implement the *self-consistent* coupling between electrostatics and transport.

C. Linearized equations

Applying a time-harmonic signal at the gate terminal $\tilde{v}_g(t) = v_0 cos(\omega t)$ of small amplitude v_0 and frequency ω, we seek a potential response of the form $U(\mathbf{r},t) = V(\mathbf{r},\omega)cos(\omega t)$, which reads in the energy domain

$$U(E) = \frac{1}{2}V(\mathbf{r},\omega)[\delta(E+\hbar\omega)+\delta(E-\hbar\omega)] \tag{16}$$

Keeping only terms to linear order in V, the ac transport-Poisson equations take the form

$$G^{\gamma}(E^+,E) = 2\pi G_0^{\gamma}(E)\delta(\hbar\omega) + \frac{1}{2}G_0^{\gamma}(E^+)V(\omega)G_0^{\gamma}(E) \tag{17}$$

$$G^{<}(E^+,E) = 2\pi G_0^{<}(E)\delta(\hbar\omega) + \frac{1}{2}G_0^{<}(E^+)V(\omega)G_0^{a}(E) + \frac{1}{2}G_0^{r}(E^+)V(\omega)G_0^{<}(E) \tag{18}$$

$$\rho(\omega) = ie\int\frac{dE}{2\pi}G^{<}(E^+,E) \tag{19}$$

$$-\rho(\mathbf{r},\omega) = \nabla[\epsilon(\mathbf{r})\nabla V(\mathbf{r},\omega)] \tag{20}$$

D. ac Response Functions: Current and Conductance

The set of Eqs. (17)-(20) obtained help determine the frequency-dependent Green functions, which can be used to obtain ac response functions. However, an additional layer of complexity is introduced by the fact that under time-dependent conditions, the total ac current I_α is not entirely determined by particle current but has, in general, contributions from the displacement current as well.

1. *Particle current $I_\alpha^p(\omega)$*

The particle current through terminal α is determined by the dynamic change in the particle density at the given terminal.

$$I_\alpha^p(t) = -e\frac{d}{dt}\langle\hat{N}_\alpha(t)\rangle = -e\frac{d}{dt}\sum_k\langle\hat{c}_{k\alpha}^{\dagger}(t)\hat{c}_{k\alpha}(t)\rangle \tag{21}$$

Making use of the fermionic anti-commutator relations and the Heisenberg equation of motion for operators $\dot{\mathcal{O}} = \frac{i}{\hbar}[H,\mathcal{O}]$ with H being the total system Hamiltonian, the particle current is given by

$$I_p^{\alpha}(t) = \frac{e}{\hbar}\,\mathrm{Tr}\int dt'[G^r(t,t')\Sigma_\alpha^<(t',t) - \Sigma_\alpha^<(t,t')G^a(t',t) + G^<(t,t')\Sigma_\alpha^a(t',t) - \Sigma_\alpha^r(t,t')G^<(t',t)] \tag{22}$$

The corresponding energy-domain representation is given by

$$I_\alpha^p(\omega) = \frac{e}{\hbar}\mathrm{Tr}\int dE[G^<(E^+,E)\Sigma_\alpha^a(E) - \Sigma_\alpha^r(E^+)G^<(E^+,E) + G^r(E^+,E)\Sigma_\alpha^<(E) - \Sigma_\alpha^<(E^+)G^a(E^+,E)] \tag{23}$$

One can now derive the dynamic conductance by expanding $\Sigma_\alpha^<$ and $G^<$ to linear order in the terminal voltage v_β, and utilizing Eqs.(17) and (18) to substitute for $G^{\gamma}(E^+,E)$ and $G^<(E+,E)$. Inserting all relevant terms in Eq. (23), the frequency-dependent particle current is given by

$$\begin{aligned} I_\alpha^p(\omega) = \frac{1}{2}\frac{e^2}{h}\sum_\beta\mathrm{Tr}\int dE[&\{G_0^{r,+}V(\omega)G_0^r\tilde{\Sigma}_\beta^< \\ &- \tilde{\Sigma}_\beta^{<,+}G_0^{a,+}V(\omega)G_0^a\}\delta_{\alpha\beta} + \tilde{G}_{0,\beta}^{<,+}V(\omega)G_0^a\Sigma_\alpha^a \\ &+ G_0^{r,+}V(\omega)\tilde{G}_{0,\beta}^{<}\Sigma_\alpha^a - \Sigma_\alpha^{r,+}\tilde{G}_{0,\beta}^{<,+}V(\omega)G_0^a \\ &- \Sigma_\alpha^{r,+}G_0^{r,+}V(\omega)G_{0,\beta}^{<}]v_\beta \end{aligned} \tag{24}$$

By definition, the tensor prefactor that relates the terminal current I_α with the applied bias v_β in Eq. (24) is the *ac linear response* particle conductance, $g_{\alpha\beta}^p$.

2. *Displacement current $I^d(\omega)$*

Under time-dependent conditions, the sum-rules for particle conductance ($\Sigma_\alpha g_{\alpha\beta} = 0$ and $\Sigma_\beta g_{\alpha\beta} = 0$) does not hold in general since the displacement current present under ac conditions is not taken into account in this formulation.

Starting from the charge continuity equation, $\partial_t\rho + \nabla\cdot\mathbf{j}^p = 0$, and integrating over volume, we obtain Kirchoff's current law under ac conditions: $I^d(t) + \Sigma_\alpha I_\alpha^p = 0$. I_α^p represents the conventional particle current through terminal α, and can be associated with a particle conductance through $I_\alpha^p = \Sigma_\beta g_{\alpha\beta}^p v_\beta$ with v_β being the voltage at terminal β. The displacement current $I^d(t) = \partial_t Q(t)$ accounts for the dynamic change in the total charge and is nonzero under time-dependent conditions.

In order to obtain an expression for the total conductance defined by $I_\alpha = \Sigma_\beta g_{\alpha\beta}v_\beta$ one needs to know how the current is split between the particle and displacement currents at each terminal. While we do have an expression for the particle current I_α^p at any given terminal α from Eq. (24), the same is not true for I^d since only the *total* rather than the *terminal* displacement current is known. Two *ansätze* have been presented for the formulation of terminal and total displacement current: $I_\alpha = I_\alpha^p + A_\alpha I^d$ and $I^d = \Sigma_\beta g_\beta^d v_\beta$, where g_β^d defines the *displacement conductance*, which then specifies a total conductance: $g_{\alpha\beta} = g_{\alpha\beta}^p + A_\alpha g_\beta^d$. The partitioning factor A_α can be determined from the sum-rules $\Sigma_\alpha g_{\alpha\beta} = \Sigma_\beta g_{\alpha\beta} = 0$, so that the total conductance is

given by

$$g_{\alpha\beta} = g^p_{\alpha\beta} - \frac{\sum_\gamma g^p_{\alpha\gamma}}{\sum_\gamma g^d_\gamma} g^d_\beta \tag{25}$$

and constiutes a $(N \times N)$ matrix for a system with N terminals, in general.

III. FLOQUET-NEGF FORMALISM

The Hamiltonian of the entire system can be defined as per Eq. (1). The device Hamiltonian, in particular, is given by

$$H_d(t) = \sum_{ij} h_{ij}(t) \hat{c}^\dagger_i \hat{c}_j \tag{26}$$

where $\hat{c}^\dagger_i$ ($\hat{c}_i$) denotes the Fermionic creation (annihilation) operator in many-body space and $h_{ij}(t) = h_{ij}(t+T)$ represents a periodic one-body Hamiltonian, which implies that the device Hamiltonian is time-periodic too with $H_d(t) = H_d(t+T)$.

The retarded Green function, G^r, satisfies the equation of motion (EOM) given by

$$[i\partial_{t'} - H(t')]\, G^r(t', t) = \delta(t' - t) \tag{27}$$

while the lesser Green function, $G^<$, satisfies the Keldysh relation

$$G^<(t', t) = \int dt_1\ dt_2\ G^r(t', t_1)\Sigma_c(t_1 - t_2)G^a(t_2, t) \tag{28}$$

where $\Sigma^<_c = \sum_{\alpha=s,d} \Sigma^<_\alpha$ refers to the contact self-energy.

Since H_d in H is time dependent, G propagators depend on two times; unlike Σ_c or $g_{L/R}$ they are not Fourier diagonal. Instead, the steady state condition for the propagators is given by

$$G(t', t) = G(t' + \frac{2\pi}{\omega_0}, t + \frac{2\pi}{\omega_0}) \tag{29}$$

Following up on Eq. (29), we can expand the system's G as a *Fourier transform* in $t' - t$ and a *Fourier series* in t as shown:

$$G(t', t) = \sum_n e^{-in\omega_0 t} \int_{-\infty}^{+\infty} \frac{dE}{2\pi} e^{-iE(t'-t)} G_n(E) \tag{30}$$

Given such a decomposition of G, we need to express the EOM in terms of the harmonics $G_n(E)$. In order to so, we redefine $G_n(E)$ (where E is unbounded) in terms of the quasienergy $\tilde{E} \in [0, \hbar\omega_0]$, i.e. $E = \tilde{E} + \hbar\omega_0$ such that $G_{mn}(\tilde{E}) = G_{m-n}(\tilde{E} + m\omega_0)$. This has the advantage that the EOM translates to a matrix equation analogous to that of a static system in Fourier space

$$\sum_m (\tilde{E} + n'\omega_0 - H_{n'm}) G^r_{mn}(\tilde{E}) = \delta_{n'n} \tag{31}$$

where $H_{n'n} = \int dt e^{i(n'-n)t} H(t)$. Note that we have essentially transformed a time-dependent Schrödinger's equation with a time-periodic Hamiltonian to a time-independent equation at the cost of an increment in the dimensionality of the basis set. This is known as the Floquet description of the steady-state dynamics in terms of sidebands, which appear formally as a new quantum number n. Time-dependent portions of $H(t)$ act as a coupling between different sidebands. The effective Hamiltonian for the nth sideband is the static portion of $H(t)$, shifted by $-n\omega_0$. One, therefore, sometimes defines the Floquet 'Hamiltonian' of the device region as

$$\mathbf{H}_{d_{nm}} = H_{d_{nm}} - n\omega_0\delta_{nm} \tag{32}$$

whereas before, $H_{d_{n'n}} = \int dt e^{i(n'-n)t} H_d(t)$. Likewise, one may define the Floquet self-energies as

$$\mathbf{\Sigma}_{c_{nm}}(\tilde{E}) = \delta_{nm}\Sigma_c(\tilde{E} + n\omega_0) \tag{33}$$

The Floquet EOM for $G^r_{nm}(\tilde{E})$ can be solved similar to the case of a static system. Within the device region of the system, we have

$$\mathbf{G}^r(\tilde{E}) = [\tilde{E} - \mathbf{h}_S - \mathbf{\Sigma}^r_c(\tilde{E})]^{-1} \tag{34}$$

Boldface denotes the sideband structure implicit in all the above matrices. Similarly, the Keldysh relation takes the simple form

$$\mathbf{G}^<(\tilde{E}) = \mathbf{G}^r(\tilde{E})\mathbf{\Sigma}_c(\tilde{E})\mathbf{G}^a(\tilde{E}) \tag{35}$$

IV. CONCLUSION

In theory, the ac-NEGF and Floquet-NEGF techniques are considered to be equally suitable for studying the behaviour of systems driven out of equilibrium by time-dependent fields. However, in practice, the particular choice depends on the specific problem being studied: for example, in the articles reviewed, the ac-NEGF technique was considered more suitable for studying the behaviour of nanotransistors, while the Floquet-NEGF method was the preferred technique for explaining the unique behaviour of superconductor-graphene-superconductor (SGS) junctions. Theoretically, the Floquet-NEGF formalism essentially eliminates the time-dependence of a system by transforming a time-dependent Schrödinger's equation with a time-periodic Hamiltonian to a time-independent equation at the cost of an increment in the dimensionality of the basis set, which appears as *Floquet quasi-levels*. In contrast, the ac-NEGF formalism retains the time dependence as well as the dimensionality of the system.

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Appendix A: Double-time Fourier Transform

The double-time Fourier transform is defined as

$$F(E,E^{'}) = \int dt\, dt^{'}\, e^{iEt/\hbar} e^{-iE^{'}t^{'}/\hbar} F(t,t^{'})$$

$$F(t,t^{'}) = \int \frac{dE}{2\pi}\frac{dE^{'}}{2\pi} e^{-iEt/\hbar} e^{iE^{'}t^{'}/\hbar} F(E,E^{'}) \quad \text{(A1)}$$

An example of such a domain transformation would be Eq. (9) where

$$\begin{aligned}\Sigma^{\gamma}(E,E^{'}) &= \int dt\, dt^{'} e^{iEt/\hbar} e^{-iE^{'}t^{'}/\hbar}\, \Sigma^{\gamma}(t,t^{'}) \\ &= \int\int dt\, dt^{'} e^{iEt/\hbar} e^{-iE^{'}t^{'}/\hbar} U^{\mathrm{DC}} \delta(t-t^{'}) \\ &+ \int\int dt\, dt^{'} e^{iEt/\hbar} e^{-iE^{'}t^{'}/\hbar} U(t) \delta(t-t^{'}) \\ &+ \int\int dt\, dt^{'} e^{iEt/\hbar} e^{-iE^{'}t^{'}/\hbar}\, \Sigma_c^{\gamma}(t-t^{'})\end{aligned} \quad \text{(A2)}$$

Let us consider each distinct term of the transform:

$$\begin{aligned} I_1 &= \int\int dt\, dt^{'} e^{iEt/\hbar} e^{-iE^{'}t^{'}/\hbar} U^{\mathrm{DC}} \delta(t-t^{'}) \\ &= \int dt' e^{-E't'/\hbar} \int U^{\mathrm{DC}}\, \delta(t-t')\, e^{iEt/\hbar} dt \\ &= \int e^{i(E-E')t'/\hbar}\, U^{\mathrm{DC}}\, dt' = 2\pi\delta(E-E')\, U^{\mathrm{DC}} \end{aligned} \quad \text{(A3)}$$

$$\begin{aligned} I_2 &= \int\int e^{iEt/\hbar}\, e^{-iE't'/\hbar}\, U(t)\, \delta(t-t')\, dt\, dt' \\ &= \int dt'\, e^{-E't'/\hbar} \int U(t)\, \delta(t-t')\, e^{iEt/\hbar} dt \\ &= \int U(t')\, e^{i(E-E')t'/\hbar}\, dt' \\ &= U(E-E') \end{aligned} \quad \text{(A4)}$$

$$\begin{aligned} I_3 &= \int\int e^{iEt/\hbar}\, e^{-iE't'/\hbar}\, \Sigma_c^{\gamma}(t-t')\, dt\, dt' \\ &= \int dt'\, e^{-iE't'/\hbar} \int \Sigma_c^{\gamma}(t-t')\, e^{iEt/\hbar}\, dt \\ &= \int dt'\, e^{-iE't'/\hbar}\, e^{iEt'/\hbar} \int \Sigma_c^{\gamma}(t'')\, e^{iEt''/\hbar}\, dt'' \\ &= \int \Sigma_c^{\gamma}(E)\, e^{i(E-E')t'/\hbar}\, dt' \\ &= 2\pi\delta(E-E')\, \Sigma_c^{\gamma}(E) \end{aligned} \quad \text{(A5)}$$

The double-time Fourier transform of the self-energy term, cf. Eq. (8), will be a summation of the terms derived in Eqs. (A4)-(A5).

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